--- HECKLE ---

A. Equations and normalization

The relevant quantities are normalized as

$$\begin{aligned} m &= \widetilde{m}\, m_p \\ q &= \widetilde{q}\, e \\ N &= \widetilde{N}\, N_o \\ B &= \widetilde{B}\, B_0 \end{aligned}$$

m_p is the proton mass, e the elementary charge, and n_0 et B_0 the standard particle density and magnetic field. Hence

$$\begin{aligned} v &= \widetilde{v}\, V_A \\ t &= \widetilde{t}\, \Omega_C^{-1} \\ l &= \widetilde{l}\, c\Omega_P^{-1} \\ E &= \widetilde{E}\, V_A B_0 \end{aligned}$$

where Ω_C and Ω_P are the proton cyclotron pulsation and proton plasma pulsation, respectively. Hence, normalized equations (omitting tildas) are

$$\begin{aligned} d_t \mathbf{x}_{s,h} &= \mathbf{v}_{s,h} \\ d_t \mathbf{v}_{s,h} &= q_s/m_s(\mathbf{E} + \mathbf{v}_{s,h} \times \mathbf{B} - \eta \mathbf{J}) \\ N(\mathbf{x}) &= \Sigma_{s,h} q_s S(\mathbf{x} - \mathbf{x}_{s,h}) \\ \mathbf{V}(\mathbf{x}) &= \Sigma_{s,h} \mathbf{v}_{s,h} S(\mathbf{x} - \mathbf{x}_{s,h}) / \Sigma_{s,h} S(\mathbf{x} - \mathbf{x}_{s,h}) \\ \partial_t \mathbf{B} &= -\boldsymbol{\nabla} \times \mathbf{E} \\ \mathbf{J} &= \boldsymbol{\nabla} \times \mathbf{B} \\ \mathbf{E} &= -\mathbf{V} \times \mathbf{B} + N^{-1}(\mathbf{J} \times \mathbf{B} - \boldsymbol{\nabla}.\mathbf{P}_e) + \eta \mathbf{J} \end{aligned}$$

where $\mathbf{V}$ is the fluid velocity of all particles except electrons, N the charge density of all particles except electrons, $\mathbf{P}_e$ the electron pressure tensor, s index standing for the specie of particle and h index for the index of the particle. $S(\mathbf{x})$ is the first order shape factor.

The electron pressure tensor $\mathbf{P}_e$ can be either isotherm ($P_e = NT_e$) or adiabatic. In both cases, the tensor is diagonal, all terms being equal with

$$\partial_t = -\mathbf{u}.\boldsymbol{\nabla} P_e - \gamma P_e \boldsymbol{\nabla}.\mathbf{u}$$

$\mathbf{u}$ being the electron flow ($\mathbf{u} = \mathbf{V} - \mathbf{J}/N$) and $\gamma = 5/3$.

if considering the electron mass, the Ohm's law is changed to

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + N^{-1}(\mathbf{J} \times \mathbf{B} - \boldsymbol{\nabla}.\mathbf{P}_e) + \eta \mathbf{J} - \frac{m_e}{e}\frac{d\mathbf{J}}{dt}$$

instead of considering the previous set of equations with this Ohm's law, we introduce pseudo-magnetic field $\mathbf{B}^\star$ and pseudo-electric field $\mathbf{E}^\star$.

The Maxwell equations governing the evolution of $\mathbf{B}^\star$ and $\mathbf{E}^\star$ are hence

$$\begin{aligned}
\partial_t \mathbf{B}^\star &= -\boldsymbol{\nabla} \times \mathbf{E}^\star \\
\mathbf{E}^\star &= -\mathbf{V} \times \mathbf{B} + N^{-1}(\mathbf{J} \times \mathbf{B}^\star - \boldsymbol{\nabla}.\mathbf{P}_e) + \eta \mathbf{J} \\
\mathbf{B}^\star &= (1 - \delta_e^2 \nabla^2)\mathbf{B}
\end{aligned}$$

The last equation being inverted using `petsc` library.

B. Definitions of normalized quantities [1]

Magnetic pressure & energy :

$$E_B = \frac{B^2}{2}$$

Generalized momentum :

$$\mathbf{p}_s = m_s \mathbf{v}_s + q_s \mathbf{A}$$

Kinetic pressure :

$$P_s = n_s m_s V_{Ts}^2 = n_s T_s$$

Thermic energy :

$$E_{Ts} = \frac{3}{2} n_s T_s = \left\langle \frac{n_s m_s v_s^2}{2} \right\rangle = \frac{3}{2} n_s m_s V_{Ts}^2$$

Plasma Beta :

$$\beta = \sum_s \beta_s = \sum_s \frac{2 n_s m_s V_{Ts}^2}{B^2} = \sum_s \frac{2 n_s T_s}{B^2}$$

Thermal Larmor radius :

$$\rho_{Ls} = \frac{2 m_s V_{Ts}}{B} = \frac{2}{B} \left(\frac{T_s}{m_s} \right)^{1/2} = \left(\frac{2 \beta_s}{n_s m_s} \right)^{1/2}$$

Thermal velocity :

$$V_{Ts} = \left(\frac{T_s}{m_s} \right)^{1/2} = \left(\frac{\beta_s B^2}{2 n_s m_s} \right)^{1/2}$$

Alfven velocity :

$$V_A = \frac{B}{n^{1/2}} = \left(\frac{2 \sum_s n_s m_s V_{Ts}^2}{\beta \sum_s n_s} \right)^{1/2} = \left(\frac{2 \sum_s n_s T_s}{\beta \sum_s n_s} \right)^{1/2}$$

1. in direction i (standing for x, y and z), one define the thermal velovity $V_{Ti}^2 = \langle V_i^2 \rangle / 2 = k_B T_i / m$. For isotrop and gyrotrop plasma, $V_T^2 = V_{Tx}^2 = V_{Ty}^2 = V_{Tz}^2$

C. "Buneman" pusher (Boris 1970)

The main equation is

$$\frac{\mathbf{v}_{n+1/2} - \mathbf{v}_{n-1/2}}{\Delta t} = \frac{q}{m}\left[\mathbf{E}_n + \left(\frac{\mathbf{v}_{n+1/2} + \mathbf{v}_{n-1/2}}{2}\right) \times \mathbf{B}_n\right]$$

Defining

$$\mathbf{v}_{n-1/2} = \mathbf{v}^- - \frac{q\mathbf{E}_n}{m}\frac{\Delta t}{2} \qquad , \qquad \mathbf{v}_{n+1/2} = \mathbf{v}^+ + \frac{q\mathbf{E}_n}{m}\frac{\Delta t}{2}$$

Hence, the equation to solve is

$$\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} = \frac{q}{2m}(\mathbf{v}^+ + \mathbf{v}^-) \times \mathbf{B}_n$$

Θ is the rotation angle between $\mathbf{v}^+$ and $\mathbf{v}^-$, $\left|\tan\frac{\Theta}{2}\right| = \frac{|\mathbf{v}^+ - \mathbf{v}^-|}{|\mathbf{v}^+ + \mathbf{v}^-|} = t = \frac{qB_n}{m}\frac{\Delta t}{2}$

Defining the intermediate v' quantity as

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t} \qquad , \qquad \mathbf{v}^+ = \mathbf{v}^- + \mathbf{v}' \times \mathbf{s}$$

To get $|\mathbf{v}^+| = |\mathbf{v}^-|$, one needs $s = \dfrac{2}{1+t^2}$.

Lets summerize :

$$F = \frac{q\Delta t}{2m} \qquad , \qquad G = \frac{2}{1 + B_n^2 F^2}$$

$$\begin{aligned}
\mathbf{s} &= \mathbf{v} + F\mathbf{E}_n \\
\mathbf{u} &= \mathbf{s} + F(\mathbf{s} \times \mathbf{B}_n) \\
\mathbf{v} &= \mathbf{s} + G(\mathbf{u} \times \mathbf{B}_n) + F\mathbf{E}_n
\end{aligned}$$

D. Algorithm - basic version

PREDICTOR :

- $\mathbf{v}_{n+1/2} = \mathbf{v}_{n-1/2} + \frac{q\Delta t}{m_p}\left[\mathbf{E}_n + \frac{\mathbf{v}_{n+1/2} + \mathbf{v}_{n-1/2}}{2} \times \mathbf{B}_n\right]$
- $\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \mathbf{v}_{n+1/2}$
- $\mathbf{N}_{n+1/2} = \Sigma_s q_s (S_n + S_{n+1})/2, \quad \mathbf{V}_{n+1/2} = \Sigma_s (S_n + S_{n+1})\mathbf{v}_{n+1/2}/2\mathbf{N}_{n+1/2}$
- $\mathbf{B}_{n+1/2} = \mathbf{B}_n - \frac{\Delta t}{2}\boldsymbol{\nabla} \times \mathbf{E}_n$
- $\mathbf{E}_{n+1/2} = -\mathbf{V}_{n+1/2} \times \mathbf{B}_{n+1/2} + \frac{1}{N_{n+1/2}}\left(\mathbf{J}_{n+1/2} \times \mathbf{B}_{n+1/2} - \boldsymbol{\nabla} P_{n+1/2}\right) + \eta \mathbf{J}_{n+1/2}$
- $\mathbf{E}_{n+1} = -\mathbf{E}_n + 2\mathbf{E}_{n+1/2}$
- $\mathbf{B}_{n+1} = \mathbf{B}_{n+1/2} - \frac{\Delta t}{2}\boldsymbol{\nabla} \times \mathbf{E}_{n+1}$

CORRECTOR :

- $\mathbf{v}_{n+3/2} = \mathbf{v}_{n+1/2} + \frac{q\Delta t}{m_p}\left[\mathbf{E}_{n+1} + \frac{\mathbf{v}_{n+3/2} + \mathbf{v}_{n+1/2}}{2} \times \mathbf{B}_{n+1}\right]$
- $\mathbf{x}_{n+2} = \mathbf{x}_{n+1} + \Delta t \mathbf{v}_{n+3/2}$
- $\mathbf{N}_{n+3/2} = \Sigma_s q_s (S_{n+1} + S_{n+2})/2, \quad \mathbf{V}_{n+3/2} = \Sigma_s (S_{n+1} + S_{n+2})\mathbf{v}_{n+3/2}/2\mathbf{N}_{n+3/2}$
- $\mathbf{B}_{n+3/2} = \mathbf{B}_{n+1} - \frac{\Delta t}{2}\boldsymbol{\nabla} \times \mathbf{E}_{n+1}$
- $\mathbf{E}_{n+3/2} = -\mathbf{V}_{n+3/2} \times \mathbf{B}_{n+3/2} + \frac{1}{N_{n+3/2}}\left(\mathbf{J}_{n+3/2} \times \mathbf{B}_{n+3/2} - \boldsymbol{\nabla} P_{n+3/2}\right) + \eta \mathbf{J}_{n+3/2}$
- $\mathbf{E}_{n+1} = \frac{1}{2}(\mathbf{E}_{n+1/2} + \mathbf{E}_{n+3/2})$
- $\mathbf{B}_{n+1} = \mathbf{B}_{n+1/2} - \frac{\Delta t}{2}\boldsymbol{\nabla} \times \mathbf{E}_{n+1}$

E. Algorithm - with electron inertia

PREDICTOR :

- $\mathbf{v}_{n+1/2} = \mathbf{v}_{n-1/2} + \frac{q\Delta t}{m_p}\left[\mathbf{E}_n + \frac{\mathbf{v}_{n+1/2} + \mathbf{v}_{n-1/2}}{2} \times \mathbf{B}_n\right]$
- $\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \mathbf{v}_{n+1/2}$
- $\mathbf{N}_{n+1/2} = \Sigma_s q_s (S_n + S_{n+1})/2, \quad \mathbf{V}_{n+1/2} = \Sigma_s (S_n + S_{n+1}) \mathbf{v}_{n+1/2}/2\mathbf{N}_{n+1/2}$
- $\mathbf{B}^\star_{n+1/2} = \mathbf{B}^\star_n - \frac{\Delta t}{2} \boldsymbol{\nabla} \times \mathbf{E}^\star_n,$ $\quad \mathbf{B}_{n+1/2}$ from $\mathbf{B}^\star_{n+1/2}$ using `petsc`
- $\mathbf{E}^\star_{n+1/2} = -\mathbf{V}_{n+1/2} \times \mathbf{B}_{n+1/2} + \frac{1}{N_{n+1/2}}\left(\mathbf{J}_{n+1/2} \times \mathbf{B}^\star_{n+1/2} - \boldsymbol{\nabla} P_{n+1/2}\right) + \eta \mathbf{J}_{n+1/2}$
- $\mathbf{E}_{n+1/2} = -\mathbf{V}_{n+1/2} \times \mathbf{B}_{n+1/2} + \frac{1}{N_{n+1/2}}\left(\mathbf{J}_{n+1/2} \times \mathbf{B}_{n+1/2} - \boldsymbol{\nabla} P_{n+1/2}\right) + \eta \mathbf{J}_{n+1/2}$
- $\mathbf{E}^\star_{n+1} = -\mathbf{E}^\star_n + 2\mathbf{E}^\star_{n+1/2}, \quad \mathbf{E}_{n+1} = -\mathbf{E}_n + 2\mathbf{E}_{n+1/2}$
- $\mathbf{B}^\star_{n+1} = \mathbf{B}^\star_{n+1/2} - \frac{\Delta t}{2} \boldsymbol{\nabla} \times \mathbf{E}^\star_{n+1}, \quad \mathbf{B}_{n+1} = \mathbf{B}_{n+1/2} - \frac{\Delta t}{2} \boldsymbol{\nabla} \times \mathbf{E}_{n+1}$

CORRECTOR :

- $\mathbf{v}_{n+3/2} = \mathbf{v}_{n+1/2} + \frac{q\Delta t}{m_p}\left[\mathbf{E}_{n+1} + \frac{\mathbf{v}_{n+3/2} + \mathbf{v}_{n+1/2}}{2} \times \mathbf{B}_{n+1}\right]$
- $\mathbf{x}_{n+2} = \mathbf{x}_{n+1} + \Delta t \mathbf{v}_{n+3/2}$
- $\mathbf{N}_{n+3/2} = \Sigma_s q_s (S_{n+1} + S_{n+2})/2, \quad \mathbf{V}_{n+3/2} = \Sigma_s (S_{n+1} + S_{n+2}) \mathbf{v}_{n+3/2}/2\mathbf{N}_{n+3/2}$
- $\mathbf{B}^\star_{n+3/2} = \mathbf{B}^\star_{n+1} - \frac{\Delta t}{2} \boldsymbol{\nabla} \times \mathbf{E}^\star_{n+1},$ $\quad \mathbf{B}_{n+3/2}$ from $\mathbf{B}^\star_{n+3/2}$ using `petsc`
- $\mathbf{E}^\star_{n+3/2} = -\mathbf{V}_{n+3/2} \times \mathbf{B}_{n+3/2} + \frac{1}{N_{n+3/2}}\left(\mathbf{J}_{n+3/2} \times \mathbf{B}^\star_{n+3/2} - \boldsymbol{\nabla} P_{n+3/2}\right) + \eta \mathbf{J}_{n+3/2}$
- $\mathbf{E}_{n+3/2} = -\mathbf{V}_{n+3/2} \times \mathbf{B}_{n+3/2} + \frac{1}{N_{n+3/2}}\left(\mathbf{J}_{n+3/2} \times \mathbf{B}_{n+3/2} - \boldsymbol{\nabla} P_{n+3/2}\right) + \eta \mathbf{J}_{n+3/2}$
- $\mathbf{E}^\star_{n+1} = \frac{1}{2}(\mathbf{E}^\star_{n+1/2} + \mathbf{E}^\star_{n+3/2}), \quad \mathbf{E}_{n+1} = \frac{1}{2}(\mathbf{E}_{n+1/2} + \mathbf{E}_{n+3/2})$
- $\mathbf{B}^\star_{n+1} = \mathbf{B}^\star_{n+1/2} - \frac{\Delta t}{2} \boldsymbol{\nabla} \times \mathbf{E}^\star_{n+1}, \quad \mathbf{B}_{n+1} = \mathbf{B}_{n+1/2} - \frac{\Delta t}{2} \boldsymbol{\nabla} \times \mathbf{E}_{n+1}$

E. Initialization

The magnetic field is initialized with the needed profile. The electric field results from the Ohm's law, and thus need not to be prescribed. The resistivity is increased near walls if not periodic. Protons (alfas...) and electrons temperature are analytically determined on the grid points.

The density profile is prescribed analytically. The weight of macro-particles is the same for all particles of specie s. The number of particles injected in each cell is linear to the local density divided by the integrated density over the whole box. To work properly, we generally use 100 particles per cells.

A drift velocity is calculated to hold $\mathbf{J} = \mathbf{\nabla} \times \mathbf{B}$. We use the classical relation $J_s/T_s =$ const for each species s including electrons.

The particle velocity is determined using the Box & Muller algorithm. a and b standing for random numbers between 0 and 1 with a normal distribution, the particle velocity in direction i is

$$\sqrt{\frac{-2\ln(a)T_{si}}{m_s}} cos(2\pi b)$$

F. Grids definitions

This code uses a simple grid representation (no Yee lattice). There is 2 grids, shifted from a half grid size. They are called `G1` and `G2`.

In one direction (both X Y and Z directions are equivalents), lets call L the size of the domain, N the number of grid cells and Δ the grid size. Of course,

$$\Delta = \frac{L}{N}$$

The `G1` grid has $N+1$ grid points associated to N cells. Grid point labelled 0 is located at $X = 0$ and grid point labelled N is located at $X = L$.

The `G2` grid has $N+2$ grid points associated to $N+1$ cells. Grid point labelled 0 is located at $X = -\Delta/2$, and grid point labelled $N+1$ is located at $X = L + \Delta/2$.

Only the magnetic field is defined on `G1`. All others quantities (electric field, electron pressure and temperature, density, fluid density, current density) are defined on `G2`.

This choice is of course motivated by the centered form of the Maxwell-Faraday equation and the leap-frog scheme to push the particles. This results in an interpollation for the electron pressure tensor when integrating the Ohm's law.

G. Boundary conditions

Because of the definition of the 2 grids, the magnetic field results from the shape of the electric field ; only the electric field needs a boundary conditions when the code is non-periodic.

Calling N the normal direction to the boundary, and T the tangential direction, We set for the electric field

$$d_N E_N = 0, \mathbf{E}_T = 0$$

For the density, it is simply $d_N n = 0$.

For the current density and fluid velocity, we use fluid conditions to keep the plasma in the domain, and thus anihilate the flux,

$$J_N = 0, d_N \mathbf{J}_T = 0$$

To limit wave reflections at the boundaries of the domain, we set a small resistivity, increased near the walls : multiplied by 5 two grid points before the limit of the domain, multiplied by 25 one grid point before the limit of the domain, and multiplied by 125 on the boundary of the domain.

H. Constraints on the code

The parameters used in classical run for $\beta = 1$ are $\Delta L = 0.4$ and $\Delta t = 0.005$.

Grid size : it has to resolve correctly the cyclotron turn of particles. If this value is too large, any bulk velocity in perpendicular direction will be converted in velocity of gyromotion (perpendicular heating). Take at least 3 grid size for the thermal larmor radius.

Time step : it has to satisfy the CFL condition for the faster mode : generally the whistler mode (at least in parallel direction). For this mode, $\omega \propto k^2$, meaning that time step has to evolve as the square of the grid size... The CFL associated to particle velocity is far less constraining.

There is no clear lower limit for the grid size, except that at one point it will cost to much in time step.

As there is no Maxwell-Gauss equation associated to neutrality, the plasma pulsation is not resolved.

A smooth is used for the moments associated to particles : density and fluid velocity. If not, the energy conservation is generally better, but the code can turn instable, generally because the density gets too low at some given points. Yet, the only clear acceptable way to manage this is to feed the simulation to prevent density holes.

A small resistivity is also used. It is supposed to have some nice consequences on the stability... We use a value of 0.0001, but its role is note that clear.